PMM U.S.S.R.,Vol.52,No.4,pp.458-460,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

NON-STATIONARY FLOWS IN CHANNELS WITH PERMEABLE WALLS*

N.M. BARASHKOV and F.F. SPIRIDONOV

A flow of inviscid fluid in a plane or axisymmetric semi-infinite channel is considered. One wall of the channel is permeable, and injection or suction of fluid takes place through the other wall at constant lengthwise intensity, while the wall itself moves according to a prescribed law. The flow is assumed to be turbulent. An equation for the stream function is obtained, and a relation connecting the law of motion of the wall with the intensity of injection (suction) is chosen such that the equation has a selfsimilar solution. An approximate representation for the selfsimilar solution is found, and viscous drag in the motions at large Reynolds numbers is estimated. The process of constructing an approximate solution when the wall moves slowly is discussed.

An approximate solution of the problem of stationary flows in channels with permeable walls, valid over a whole range of variation in the characteristic Reynolds number R, was constructed in /1/. It was shown that by virtue of the specific features of the problem (no slippage at the walls), the profiles of the velocity vector components vary insignificantly as R varies. A method given earlier in /1/ is used here to obtain self-similar solution of non-stationary problems in the limiting case when $R \to \infty$.

1. Consider a flow of incompressible fluid in a plane $(\nu = 0)$ or axisymmetric cylindrical $(\nu = 1)$ channel of half-width a, varying with time t according to the law $a = a(t)$. Intense injection or suction occurs through the channel wall in a direction normal to it, at a rate $q = q(t)$ constant along the permeable wall, and such that the characteristic Reynolds number $R = |\rho q a/\mu| \to \infty$. The motion of the fluid is considered in a Cartesian or cylindrical coordinate system (z, y), where the z axis coincides with the plane (axis) of symmetry of the flow and y is orthogonal to it.

The flow is described by the following system of equations:

$$\frac{\partial \omega}{\partial t} + w\frac{\partial \omega}{\partial z} + y^\nu v \frac{\partial}{\partial y}\left(\frac{\omega}{y^\nu}\right) = 0, \quad \omega = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \tag{1.1}$$

where ω is the vorticity, and w, v are the components of velocity vector along the z and y axis respectively, and are connected with the stream function ψ by the relations

$$w = \frac{1}{y^\nu}\frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{y^\nu}\frac{\partial \psi}{\partial z} \tag{1.2}$$

The boundary and initial conditions are

$$\begin{aligned} &y = 0,\ v = 0;\ y = a,\ v = q + da/dt,\ w = 0 \\ &z = 0,\ \partial w/\partial z = 0 = \partial v/\partial z \\ &t = 0,\ a = a_0,\ q = q_0 \end{aligned} \tag{1.3}$$

Let us introduce the dimensionless variables

$$z^\circ = z/a,\ y^\circ = y/a,\ w^\circ = w/q,\ v^\circ = v/q$$

Then Eq.(1.1) will be replaced by

$$-\omega^\circ \frac{\partial}{\partial t}\frac{a}{q} + w^\circ \frac{\partial \omega^\circ}{\partial z^\circ} + y^{\circ\nu} v^\circ \frac{\partial}{\partial y^\circ}\frac{\omega^\circ}{y^{\circ\nu}} = 0 \tag{1.4}$$

We shall seek the selfsimilar solutions of Eq.(1.4), and this demand will establish the need to satisfy the relation

*Prikl.Matem.Mekhan.,52,4,590-593,1988

$$\partial (a/q)/\partial t = \alpha = \mathrm{const}$$

which imposes on the laws of variation of $a(t)$ and $q(t)$ a constraint of the following form:

$$a(t) = q(t)\ (\alpha t + \beta)$$

Here β is a constant which can be found from the initial conditions of the problem.

Let us consider a stream function of the form

$$\psi^\circ = z^\circ F(\eta),\quad \eta = y^{\circ\, 1+\nu} \tag{1.5}$$

satisfying, without loss of generality, the boundary conditions (1.3).

Substituting into (1.4) the expressions for w°, v° and ω° obtained, taking relation (1.5) into account, we obtain the following equation from the defining relations of the type (1.2):

$$(1+\nu)^{-1}\alpha F'' - F'F'' + FF''' = 0 \tag{1.6}$$

(a prime denotes differentiation with respect to the variable η).

The boundary conditions (1.3) yield the following boundary conditions for the Eq.(1.6):

$$F(0) = F'(1) = 0,\ F(1) = \pm(1-\alpha) \tag{1.7}$$

The plus sign in the last expression corresponds to suction of the fluid from the channel with $da/dt < 0$, and the minus sign corresponds to injection of the fluid with $da/dt > 0$.

Let us pass, in problem (1.6), (1.7), to a new dependent variable $\varphi(\eta) = F(\eta)/F(1)$. We obtain

$$\gamma\varphi'' - \varphi'\varphi'' + \varphi\varphi''' = 0,\ \gamma = (1+\nu)^{-1}\alpha/F(1) \tag{1.8}$$

$$\varphi(0) = \varphi'(1) = 0,\quad \varphi(1) = 1 \tag{1.9}$$

2. We shall seek an approximate solution of the boundary value problem (1.8), (1.9) using the method given in /1/. Since the solution sought is assumed (for physical reasons) to be smooth, we shall write it, taking into account the condition of symmetry of the axial component of the velocity vector, in the form of a series

$$\begin{aligned}\varphi(\eta) = {}^3/_2\eta - {}^1/_2\eta^3 + a_7(2\eta - 3\eta^3 + \eta^7) + \\ a_9(3\eta - 4\eta^3 + \eta^9) + \ldots\end{aligned} \tag{2.1}$$

Expression (2.1) satisfies the boundary conditions (1.9). Additional conditions are, however, necessary in order to obtain the values of the coefficients $a_7, a_9, \ldots$

One of these conditions

$$\varphi(1)\varphi'''(1) + \gamma\varphi''(1) = 0 \tag{2.2}$$

can be obtained directly from (1.8) and (1.9) by putting $\eta = 1$.

We obtain the second condition by integrating Eq.(1.8) over the interval of variation of η:

$$[\varphi\varphi'' - \varphi'^2 + \gamma\varphi']_0^1 = 0 \tag{2.3}$$

We can obtain a number of such conditions, but we shall limit ourselves to conditions (2.2) and (2.3), and determine only a_7, a_9 in expressions (2.1). Substituting this expression into conditions (2.2) and (2.3), we obtain

$$\begin{aligned}&a_7 = C + Da_9,\ E^2a_9^2 - 2Ga_9 - H = 0\\ &C = {}^1/_8(1+\gamma)/(8+\gamma),\ D = -2(10+\gamma)/(8+\gamma)\\ &E = -2(16+\gamma)/(8+\gamma),\ G = 2(2176 + 408\gamma + 7\gamma^2 - \\ &\quad \gamma^3)/(8+\gamma)^2\\ &H = {}^1/_{64}(16895 + 33342\gamma + 7487\gamma^2 + 448\gamma^3)/(8+\gamma)^2\end{aligned} \tag{2.4}$$

The relation $4G^2 \gg |E^2H|$ holds in the most interesting interval in practice $0 \leqslant \gamma \leqslant 1$; therefore the following expression will represent a good approximation to the smallest root of the second equation of (2.4):

$$a_9 = -\frac{H}{2G}\left(1 - \frac{E^2H}{4G^2}\right) \tag{2.5}$$

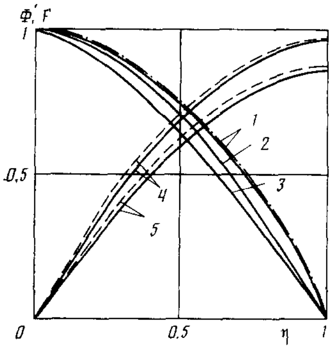
Thus relation (2.1), the first relation of (2.4) and (2.5) together yield the required solution of the problem. We can carry out an identical analysis for the case when the Reynolds number has finite values.

3. Let us compare the approximate solution obtained for $\gamma = 0$, with the exact limiting solution as $(R \to \infty)$

$$\varphi(\eta) = \sin {}^1/_2\pi\eta \tag{3.1}$$

In this case the coefficients in the approximate solution (2.1) have the form

$$a_7 = 0.05415,\ a_9 = -0.01541 \tag{3.2}$$



The first derivatives of expressions (3.1) and (2.1), (3.2) determine the corresponding profiles of the longitudinal component of the velocity vector in the channel. The figure shows these relationships (the dot-dash and solid lines *1*), as well as the analogous relationships for the values $\gamma = 0.5$ (curve *2*) and $\gamma = 1.0$ (curve *3*), $\Phi' = \varphi'/\varphi'(0)$. We see that the velocity profile in the channel changes very little over a wide range of variation in the value of the parameter γ, and differs very little from the stationary profile.

By virtue of the specific features of the problem (no tangential component of the velocity vector at the permeable wall for any value of the Reynolds number), and since the coefficient of friction at the wall

$$c_f = 2\tau_w/(\rho w_m^2), \quad \tau_w = \mu\,(\partial w/\partial y)_w$$

differs very little, as the results of /1/ for $\gamma = 0$ and $R \geqslant 100$ indicate, from its limiting value as $R \to \infty$, we find that in the present case we can analyse the change in the limiting value c_f relative to the change in γ. Here τ_w is the intensity of friction at the permeable wall, and w_m is the longitudinal velocity averaged over the transverse cross-section of the channel.

If we define the relative coefficient of friction as $c_{fr} = c_f/c_{f0}$, where the zero subscript denotes the value of c_f at $\gamma = 0$, we can easily show that $c_{fr} = \varphi''(1)/\varphi_0''(1)$. The relation $c_{fr} = c_{fr}(\gamma)$ calculated with help of the solution obtained, is practically linear: $c_{fr} = 1 - 0.32\,\gamma$. The limiting value of the coefficient of friction changes by more than a factor of 1.5 when γ varies from zero to one.

4. In conclusion we note that when $\alpha = o(1)$ a simple approximate solution of the boundary value problem (1.6), (1.7) can be obtained using the fact that although the small parameter α occurs in the boundary conditions, it does not appear in the equation as a coefficient accompanying the higher derivative. Therefore, the solution must behave in a regular manner during the passage to the limit as $\alpha \to 0$. We can seek the solution of the problem in question, satisfying the boundary conditions, in the form

$$F(\eta) = \sum_{k=0}^{\infty} c_k \sin\frac{\pi}{2}(2k+1)\,\eta \tag{4.1}$$

Confining ourselves to the first term of the series in (4.1), substituting the function $F(\eta) = c_0 \sin {}^1/_2\pi\eta$ into Eq.(1.6) and applying the collocation at the point $\eta = 0$, we can satisfy the equation formally. From the condition $F(1) = \pm(1-\alpha)$ it follows that $c_0 = \pm(1-\alpha)$, and the approximate solution of the problem has the following form for small α:

$$F(\eta) = \pm(1-\alpha)\sin {}^1/_2\pi\eta \tag{4.2}$$

When $\alpha = 0$, the solution is identical with solution (3.1). It can be shown that the modulus of the discrepancy δ in Eq.(1.6) does not exceed $\pi^2\alpha(1+\nu)^{-1}/4$, i.e. when $\alpha = 0.01$, the value of δ does not exceed 3%. The figure shows, for comparison, the solutions (2.1) and (4.2) by the solid and dashed lines respectively, for the values of the parameter $\alpha = 0.05$ (curves *4*) and $\alpha = 0.15$ (curves *5*). We see that the solutions differ very little from each other.

REFERENCES

1. MORDUCHOW M., On laminar flow through a channel or tube with injection: application of method of averages, Quart. Appl. Math. 14, 4, 1956.

Translated by L.K.